A q-OSCILLATOR WITH 'ACCIDENTAL' DEGENERACY OF ENERGY LEVELS

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We study main features of the exotic case of q-deformed oscillators (so-called Tamm-Dancoff cutoff oscillator) and find some special properties: (i) degeneracy of the energy levels $E_{n_1} = E_{n_1+1}$, $n_1 \geq 1$, at the *real value* $q = \sqrt{\frac{n_1}{n_1+2}}$ of deformation parameter, as well as the occurrence of other degeneracies $E_{n_1} = E_{n_1+k}$, for $k \geq 2$, at the corresponding values of q which depend on both n_1 and k; (ii) the position and momentum operators X and P *commute on the state* $|n_1\rangle$ if q is fixed as $q = \frac{n_1}{n_1+1}$, that implies unusual uncertainty relation; (iii) two commuting copies of the creation, annihilation, and number operators of this q-oscillator generate the corresponding q-deformation of the *non-simple* Lie algebra $su(2) \oplus u(1)$ whose nontrivial q-deformed commutation relation is: $[J_+, J_-] = 2J_0\, q^{2J_3-1}$ where $J_0 \equiv \frac{1}{2}(N_1 - N_2)$ and $J_3 \equiv \frac{1}{2}(N_1 + N_2)$.

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1. Introduction

Since the famous papers[1,2], the diverse aspects and applications of Biedenharn-Macfarlane (BM) q-deformed oscillators have become a very popular subject. This same can be said about the somewhat earlier introduced and slightly more simple Arik-Coon (AC) version[3] of q-deformed oscillator. The principal difference between these two versions is that unlike the latter, the former one admits not only real, but also the phase-like complex values of the deformation parameter q, and this has important consequences. For instance, a possibility of 'accidental' degeneracies does appear in case of q being a root of unity, popular values for the BM type q-oscillator. Namely, for $q = \exp(\frac{\mathrm{i}\pi}{2n+2})$ the following two neighboring energy levels coincide: $E_{n+1} = E_n$. This and other coincidences lead to a kind of periodicity and naturally make the corresponding phase space both discrete and finite[4].

The question arises: *is it possible that analogous property of 'accidental' degeneracy may occur at real value(s)* of the q-parameter for, maybe, some other type of q-deformed oscillator? Say, is it possible that fixation of the deformation parameter by some real value(s), e.g. $q = \sqrt{0.98}$ or $q = 1.02^{-1/2}$, does provide degeneracy of respective energy levels? The goal of this note is to present an analysis of such an exotic q-oscillator and to show it possesses the already mentioned degeneracy properties, and also a couple of other unusual properties. This type of q-oscillator has first appeared in Ref. 5, 6 where it was called the 'Tamm-Dancoff cutoff' deformed oscillator. Representation-theoretic, unified view, and generalized (f-oscillator) aspects of the q-oscillator algebra were studied in Ref. 7, 8, 9. Our interest to this q-oscillator has emerged in the course of our recent study of possible application[10] of the (set of) two-parameter q,p-deformed oscillators (those include the already mentioned BM and AC q-oscillators as the two particular $p = q^{-1}$ and $p = 1$ limiting cases). Namely, the q,p-deformed oscillators have been utilized in the framework of the model of two-parameter deformed q,p-Bose gas model, for which the explicit formulas for (intercepts of) general n-particle momentum correlation functions have been obtained[11]. Note that these results generalize previously known formulas for two-particle correlations in the AC and BM versions of q-Bose gas model.

2. Setup on the q-oscillator

The q-deformed oscillator of our main interest here is defined by the relations:

$$aa^\dagger - qa^\dagger a = q^N \ , \tag{1}$$

$$[N, a] = -a \ , \qquad [N, a^\dagger] = a^\dagger \ . \tag{2}$$

To state the crucially important relation between the number operator N, on one side, and the operators $a^\dagger a$ or $aa^\dagger$, on the other, we appeal to the well-known generalized two-parameter family[12] of p, q-deformed oscillators defined by the relations

$$AA^\dagger - q\ A^\dagger A = p^{\mathcal{N}} \ , \qquad\qquad AA^\dagger - p\ A^\dagger A = q^{\mathcal{N}} \ , \tag{3}$$

supplemented with two relations involving $\mathcal{N}$ (completely analogous to (2)).

From the pair of relations (3) invariant under $q \leftrightarrow p$, the formulas follow:

$$A^\dagger A = [\![\mathcal{N}]\!]_{qp} \ , \qquad AA^\dagger = [\![\mathcal{N}+1]\!]_{qp} \ , \qquad [\![X]\!]_{q,p} \equiv \frac{q^X - p^X}{q - p} \ . \tag{4}$$

At $p = 1$ this two-parameter system reduces to the AC-type[3] of q-bosons, while putting $p = q^{-1}$ yields the other already mentioned case[1,2] of q-oscillators.

At $p = q$, each of the relations in (3) turns into the single relation (1) what precludes connecting of $A^\dagger A$ or $AA^\dagger$ immediately with $\mathcal{N}$. Besides, the q,p-bracket in (4) becomes undefined if $p = q$, for X being either an operator or a number. Only for non-negative integer k it makes sense, since

$$[\![k]\!]_{q,p} = \frac{q^k - p^k}{q - p} = \sum_{r=0}^{k-1} q^{k-1-r} p^r = q^{k-1} \sum_{r=0}^{k-1} q^{-r} p^r \quad \stackrel{p \to q}{\Longrightarrow} \quad kq^{k-1} \ . \tag{5}$$

Then, let us take the analogue of the last part of relation (5) as the (at first, formal) definition of the new q-bracket in all cases, both for operators and for numbers:

$$\{X\}_q \equiv X q^{X-1} \ . \tag{6}$$

To find more strict justification for the definition (6), let us consider the one-parameter subfamily of q,p-oscillators (3)-(4). Namely, let $p = q^m$ with real m:

$$AA^\dagger - q\ A^\dagger A = q^{m\mathcal{N}} \qquad AA^\dagger - q^m\ A^\dagger A = q^{\mathcal{N}} \ , \tag{7}$$

$$A^\dagger A = [\![\mathcal{N}]\!]_{q,q^m} \ , \qquad AA^\dagger = [\![\mathcal{N}+1]\!]_{q,q^m} \ .$$

In the limit of $m \to 1$ this reduces to the q-oscillator (1), so we consider

$$\lim_{m\to 1}[\![X]\!]_{q,q^m} \equiv \lim_{m\to 1}\frac{q^X - q^{mX}}{q - q^m} \ . \tag{8}$$

It is useful to merely put $\widetilde{q} \equiv q^{m-1}$, so that $\widetilde{q} \overset{m\to 1}{\Longrightarrow} 1$. Then,

$$[\![X]\!]_{q,q^m} = q^{X-1}\ \frac{\widetilde{q}^X - 1}{\widetilde{q} - 1} \ . \tag{9}$$

So we define

$$\{X\}_q \equiv q^{X-1} \cdot \lim_{\widetilde{q}\to 1}\left(\frac{\widetilde{q}^X - 1}{\widetilde{q} - 1}\right) = q^{X-1}\ X \tag{10}$$

getting the same as in (6) (we have used the L'Hospital rule in order to resolve the uncertainty).

Taking into account the formulas

$$a^\dagger a = \{N\}_q \equiv N q^{N-1} \ , \qquad aa^\dagger = \{N+1\}_q \equiv (N+1) q^N \ , \tag{11}$$

we verify that the defining relation (1) is satisfied.

Note that the q-oscillator algebra generated by a, $a^\dagger$ and N obeying the relations (1)-(2) may be given a Hopf algebra structure, with the comultiplication, antipode and counit written out as in Ref. 6, 13.

3. Energy spectrum of the q-oscillator

Let us study in some detail the properties of the q-oscillator (1). To this end we take, in analogy with usual quantum harmonic oscillator, the Hamiltonian of the q-oscillator (1) in the form

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) \tag{12}$$

and from now on put $\hbar\omega = 1$ for simplicity.

We adopt the q-analogue of the Fock space with the vacuum state $|0\rangle$. Then,

$$a|0\rangle = 0 \ , \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\{n\}_q!}}|0\rangle \ , \qquad N|n\rangle = n\ |n\rangle \ , \tag{13}$$

where $\{n\}_q! = \{n\}_q\{n-1\}_q ... \{2\}_q\{1\}_q$, curly brackets are defined in (11), and the creation/annihilation operators act by the formulas

$$a\ |n\rangle = \sqrt{nq^{n-1}}\ |n-1\rangle\ , \qquad a^{\dagger}\ |n\rangle = \sqrt{(n+1)q^{n}}\ |n+1\rangle\ . \tag{14}$$

For any real nonnegative q, the operators a and $a^{\dagger}$ are indeed conjugates of each other.

From (12)-(14), the spectrum $H|n\rangle = E_n|n\rangle$ of the Hamiltonian reads:

$$E_n = \frac{1}{2}\Big((n+1)q^n + nq^{n-1}\Big) = \frac{1}{2}\ q^n\Big(1 + n(1+q^{-1})\Big)\ . \tag{15}$$

At $q \to 1$ we recover the familiar result $E_n = n + \frac{1}{2}$ as it should. Moreover, at $n = 0$ we have $E_0 = \frac{1}{2}$ whatever is the value of q.

For any $q \neq 1$, the *spectrum is not uniformly spaced* (not equidistant). It is easy to see that if $q > 1$ the spacing $E_{n+1} - E_n$ gradually increases with growing n, so that $E_n \to \infty$ as $n \to \infty$. Different, and more interesting situation emerging at $0 < q < 1$ is the subject of our study in the next sections.

4. Various 'accidental' degeneracies $E_{n_1} = E_{n_2}$ of energy levels

From now on we deal with the values of q such that

$$0 < q < 1\ . \tag{16}$$

We have to stress that the special feature of the q-oscillator defined by the relations (1)-(2), with any fixed value of the q-deformation parameter from (16), consists in the asymptotical tending, as $n \to \infty$, of the n-th level energy to zero : $E_n \to 0^+$. For that reason the q-oscillator (1)-(2) has been named the "Tamm-Dankoff cutoff" oscillator[5,6]. In this letter we use this same term (or "TD" in short).

Remark 1. If one imposes the requirement[13] that for all the energy values the inequality $\Delta E(n) \equiv E_{n+1} - E_n > 0$ *must hold*, then the spectrum is truncated and becomes finite: the set of energy levels consists of E_n with $n = 0, 1, ..., \lfloor\frac{1+q^2}{1-q^2}\rfloor$ (here $\lfloor x\rfloor$ denotes integral part of x). We however do not require this in our paper, and consider the whole (infinite) spectrum.

The energy spectrum given by the expression (15) manifests some sorts of degeneracies, with strong dependence on the particular fixed value of q. Let us consider a number of different cases.

4.1. *Degeneracy of nearest-neighbor levels: $E_m = E_{m+1}$*

Let the q be fixed as $q = \sqrt{\frac{1}{3}}$. Then it turns out that

$$E_0 = \frac{1}{2} < E_1 = E_2 = \frac{1}{2} + \frac{1}{\sqrt{3}}\,,$$

$$E_2 > E_3 = \frac{1}{2} + \frac{2}{3\sqrt{3}}\ > E_4 = \frac{5}{18} + \frac{2}{3\sqrt{3}} > \frac{1}{2}\ ,$$

and for the rest (infinite set) of levels we find

$$\frac{1}{2} > E_5 = \frac{1}{2} + \frac{\sqrt{3}-2}{9} > E_6 > \ldots > E_m > E_{m+1} > \ldots > 0 \,, \qquad m = 7, 8, \ldots \,.$$

This degeneracy $E_1 = E_2$ is a particular case of the next more general property.

Proposition 1. Let the parameter q be fixed as $q = \sqrt{\frac{m}{m+2}}$ where $m \geq 1$. Then the degeneracy occurs for the following pair of the energy levels:

$$E_m = E_{m+1} \,. \tag{17}$$

This is proved by direct check. Figs. 1,2,3 illustrate such a degeneracy for three particular cases, $m = 1, 2, 9$.

Note that $n = 0$ is excluded from the above series (17) since the degeneracy $E_0 = E_1$ is not possible, as this would require $q = 0$. However, yet another degeneracies with the E_0 involved does occur, see below.

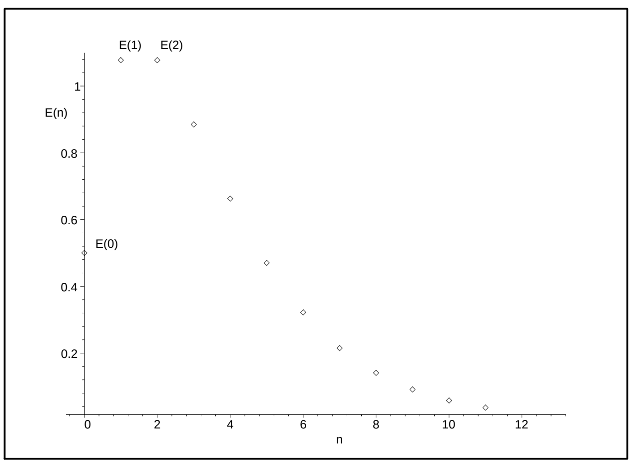

Fig. 1. Spectrum of the q-oscillator (1) at fixed $q = \sqrt{1/3}$. Observe that $E_0 = \frac{1}{2}$ and $E_1 = E_2$.

4.2. *Degeneracy of next to nearest neighbors:* $E_m = E_{m+2}$

Let the q-parameter be fixed as $q = \frac{1}{3}$. Then,

$$E_0 = \frac{1}{2} \,, \qquad E_1 = \frac{5}{6} > \frac{1}{2} \,, \qquad E_2 = E_0 = \frac{1}{2} \,, \qquad E_3 = \frac{13}{54} < \frac{1}{2} \,,$$

while for all other levels we find $\frac{1}{2} > E_m > E_{m+1}$, $m \geq 2$. This instance is nothing but a particular case of the next more general situation.

Proposition 2. Let the parameter q be fixed as

$$q = \frac{1 + \sqrt{4m^2 + 12m + 1}}{2(m+3)} \qquad \text{with} \qquad m \geq 0 \,. \tag{18}$$

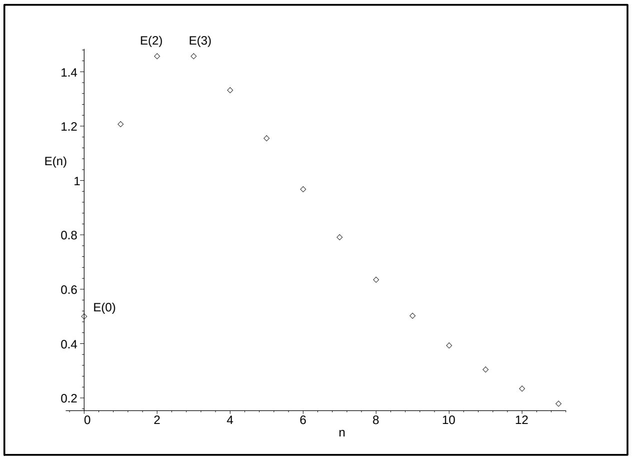

Fig. 2. Spectrum of the q-oscillator (1) at fixed $q = \sqrt{2/4}$. Observe the degeneracy $E_2 = E_3$.

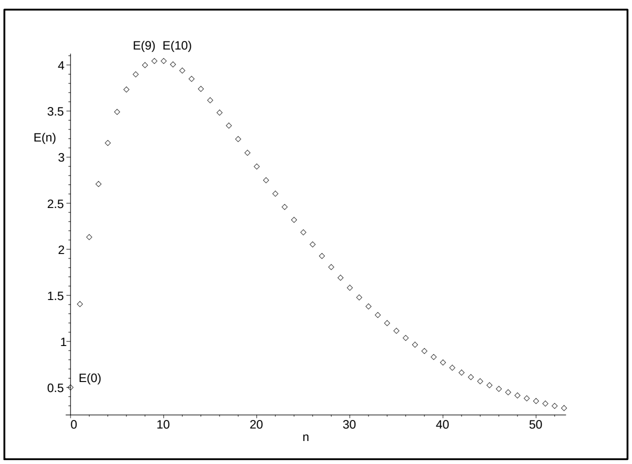

Fig. 3. Spectrum of the q-oscillator (1) at fixed $q = \sqrt{9/11}$. Observe the degeneracy $E_9 = E_{10}$.

Then among the energy levels E_n the following degeneracy occurs:

$$E_m = E_{m+2} \ . \tag{19}$$

This statement is proved by direct verification. The two particular cases $m = 0$ and $m = 5$ of the series of degeneracies (19) are shown in Fig. 4 and Fig. 5.

4.3. *Degeneracy of the type* $E_0 = E_n$

Now let us study the occurrence of yet another type of degeneracy, $E_0 = E_n$.

Proposition 3. For any integer $n = 2, 3, 4, \ldots$, there exists the appropriate value $q_n = q(n)$ that guarantees validity of the equality

$$E_0 = E_n \ . \tag{20}$$

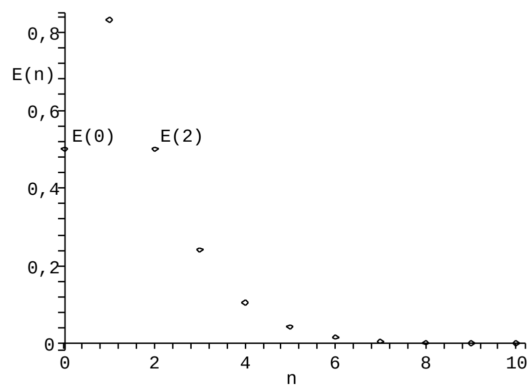

Fig. 4. Energy levels E_0, E_1,..., E_{10} of the q-oscillator (1) at $q = \frac{1}{3}$, see (18). Observe $E_0 = E_2$.

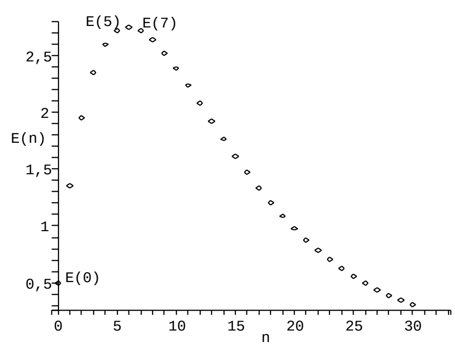

Fig. 5. Energy levels E_0, ..., E_{30} of the q-oscillator (1) at $q = \frac{1+\sqrt{161}}{16}$, see (18). Observe $E_5 = E_7$.

The proposition can be proved by graphical treatment. Suppose (20) is valid. Then

$$(n+1)q^n + nq^{n-1} = 1 \qquad\qquad (0 < q < 1) \tag{21}$$

or, equivalently, using for convenience x instead of n,

$$(1 + q^{-1})x = q^{-x} - 1 \ . \tag{22}$$

One can easily check the fact of intersection (at x *being the integers* 2, 3, 4, ...) of the curves corresponding to the linear resp. exponential functions in the l.h.s. resp. in the r.h.s. of the latter equality. Fig. 6 and Fig. 7 serve to illustrate this fact.

It is instructive to have some particular values of the q-parameter which provide the degeneracies of the type (20). To this end, we rewrite Eq. (21) as the equation for $z \equiv q^{-1}$:

$$z^m - mz - (m+1) = 0 \ , \qquad\qquad m = 2, 3, 4, ...$$

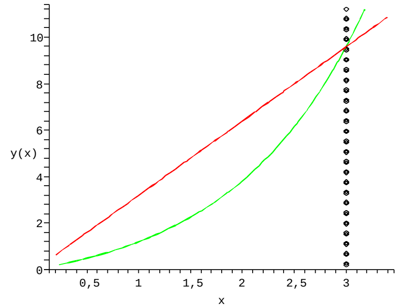

Fig. 6. Degeneracy $E_0 = E_3$ at $q = q_3 \simeq 0.45541$ from graphical solution of Eq. (22).

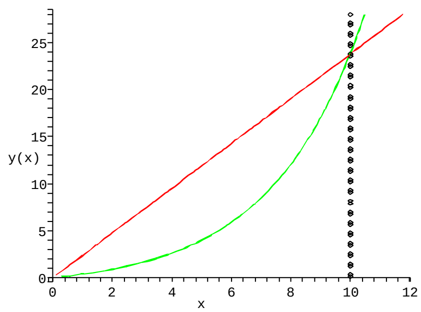

Fig. 7. Degeneracy $E_0 = E_{10}$ at $q = q_{10} \simeq 0.725405$ from graphical solution of Eq. (22).

The first three members of this series are solved exactly (in radicals), while for $m \geq 5$ the values q_m are found approximately (recall that only those solutions q_m are admitted that obey $0 < q_m < 1$). Table 1 presents a list of sample values.

Table 1. Value of q_m that gives $E_0 = E_m$.

Value of m	Value of q_m
$m = 2$	$q_2 = \frac{1}{3} \simeq 0.333333$
$m = 3$	$q_3 = \left(\sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}} \right)^{-1} \simeq 0.45541$
$m = 4$	$q_4 = 3\left(1 + \sqrt[3]{64+6\sqrt{114}} + \sqrt[3]{64-6\sqrt{114}} \right)^{-1} \simeq 0.53156446$
$m = 5$	$q_5 \simeq 0.585442$
$m = 6$	$q_6 \simeq 0.6262253$
$m = 10$	$q_{10} \simeq 0.725405$
$m = 25$	$q_{25} \simeq 0.851675$
$m = 50$	$q_{50} \simeq 0.910968$
$m = 100$	$q_{100} \simeq 0.948094$
$m = 200$	$q_{200} \simeq 0.9704016$
$m = 400$	$q_{400} \simeq 0.98340363$

4.4. *General two-fold 'accidental' degeneracy* $E_m = E_{m+k}$

Here we briefly discuss the most general form of two-fold degeneracy $E_m = E_{m+k}$. The equation for the q-parameter value $q = q(m,k)$ responsible for this fact reads $(m+k+1)q^{m+k} + (m+k)q^{m+k-1} - (m+1)q^m - mq^{m-1} = 0$, or

$$(m+k+1)\; q^{k+1} + (m+k)\; q^k - (m+1)\; q - m = 0 \; . \tag{23}$$

It can be proved that for each pair $(m,\; m+k)$ there exists such real solution $q = q(m,k)$ of (23) that $0 < q < 1$. Here we only comment on few cases of low k values. It is obvious that $k = 1$ resp. $k = 2$ correspond to the particular series of degeneracies already considered above, in subsection 4.1 resp. subsection 4.2. For the next two cases the equations to be solved are:

$$\underline{k=3} \qquad\qquad q^4 + \frac{m+3}{m+4}\, q^3 - \frac{m+1}{m+4}\, q - \frac{m}{m+4} = 0 \; , \tag{24}$$

$$\underline{k=4} \qquad\qquad q^4 - \frac{1}{m+5}\, q^3 + \frac{1}{m+5}\, q^2 - \frac{1}{m+5}\, q - \frac{m}{m+5} = 0 \tag{25}$$

where for $k = 4$ it was taken into account that the starting 5-th degree equation divides by $q + 1$. Note that the root $q = -1$ also exists in all the cases of higher even k in (23). The equations (24),(25) can be solved in radicals, what yields huge expressions. And, analogously to the above Table 1, a set of values $q = q(m,k)$ can be found numerically and as well tabulated. Finally, let us remind that the case $E_0 = E_m$, see subsec. 4.3, is obviously covered by the most general situation (23).

5. Fibonacci property of the TD q-oscillator

Fibonacci oscillators, see Ref. 15, are characterized by the property that each their energy eigenvalue E_{n+1} is uniquely determined by a linear combination of the two preceding eigenvalues:

$$E_{n+1} = \alpha\, E_n + \beta\, E_{n-1}, \qquad \alpha, \beta \in \mathbf{R}. \tag{26}$$

Taking into account the explicit formula (15) one verifies that the TD-oscillator *does possess the Fibonacci property* (26) if the coefficients are chosen as

$$\alpha = 2\, q \; , \qquad\qquad \beta = -q^2 \; . \tag{27}$$

Now it is of interest to consider the interplay "Fibonacci + degeneracy", that is, to examine how a specified version of the Fibonacci property manifests itself 'locally', i.e. for those three consecutive levels which include some degenerate pair $E_{n_1} = E_{n_2}$.

Let us first put $q = \sqrt{\frac{m-1}{m+1}}$, $\; m > 1$, which results in $E_{m-1} = E_m$. Then,

$$E_{m+1} = q(2-q)E_m \qquad \text{or} \qquad E_{m+1} = q(2-q)E_{m-1}\,, \qquad q = \sqrt{\frac{m-1}{m+1}}\,. \tag{28}$$

Likewise, if we let $q = \sqrt{\frac{m}{m+2}}$ with $m > 0$ then $E_{m+1} = E_m$. In this case we have:

$$E_{m-1} = \frac{2q-1}{q^2}\, E_m \qquad \text{or} \qquad E_{m-1} = \frac{2q-1}{q^2}\, E_{m+1}\ , \qquad q = \sqrt{\frac{m}{m+2}}\ . \tag{29}$$

Yet another situation corresponds to account of the degeneracy of next to nearest neighbors, $E_{m+2} = E_m$, for the q-parameter fixed exactly as in (18). In this case:

$$E_{m+1} = \frac{1+q^2}{2q}\, E_m \qquad \text{or} \qquad E_{m+2} = \frac{2q}{1+q^2}\, E_{m+1} \tag{30}$$

where $q = \frac{1+\sqrt{4m^2+12m+1}}{2(m+3)}$. In a similar way, other two-fold degeneracies analyzed in Section 4 can be dealt with in conjunction with the Fibonacci property (26).

6. 'Locally classical' appearance of the commutator of X and P

We use standard form for the position/momentum operators: $X \equiv \frac{1}{\sqrt{2}}(a + a^\dagger)$, $P \equiv \frac{\mathrm{i}}{\sqrt{2}}(a^\dagger - a)$. With the account of (11), the commutator of these operators is

$$-\frac{i}{\sqrt{2}}\,[X,P] = [a,a^\dagger] = (N+1)q^N - Nq^{N-1} = q^N + (q-1)Nq^{N-1}$$

or, on the basis states,

$$-\frac{i}{\sqrt{2}}\,[X,P]\,|n\rangle = \big(q^n + (q-1)nq^{n-1}\big)|n\rangle \ = q^n\big(1 + n(1-q^{-1})\big)|n\rangle\,. \tag{31}$$

If we fix the deformation parameter as the rational number $q = \frac{m}{m+1}$, then it turns out that 'locally', i.e. on the individual basis element $|n = m\rangle$, the two operators X, P behave just as the "classical" ones:

$$[X,P]\,|m\rangle = 0 \qquad\qquad \text{if} \qquad\qquad q = \frac{m}{m+1}\ . \tag{32}$$

Like in the cases of AC and BM type q-oscillators[3,1] whose $X - P$ uncertainty relations depend on both the q-parameter and the state-labeling number n (i.e., *state-dependent uncertainty relations*[1,14]), the q-deformed oscillator treated in the present letter also yields the commutator X, P, and its associated uncertainty relation, with the explicit $|n >$ dependence, see (31). However, unlike the AC q-oscillator with *real* q, the TD type q-oscillator admits for each state $|m >$ the respective special value (32) of deformation parameter which makes X and P 'classical' (commuting) on this, and only this, state. This fact has evident consequence for (the minimum of) associated uncertainty relation.

7. Quantum algebra realized with TD q-oscillators

Using two identical and independent (commuting) copies of the TD q-oscillator with the generators $a_1, a_1^\dagger$, $a_2, a_2^\dagger$, N_1, N_2 obeying (1), (2), (11) we form the generators

$$J_+ := a_1^\dagger a_2\ , \qquad J_- := a_2^\dagger a_1\ , \qquad J_0 := \frac{1}{2}\big(N_1 - N_2\big)\ , \quad J_3 := \frac{1}{2}\big(N_1 + N_2\big)\ ,$$

and find their closing into the following q-deformed algebra

$$[J_0, J_\pm] = \pm J_\pm \ , \quad [J_+, J_-] = 2J_0 \ q^{2J_3-1} \ , \quad [J_0, J_3] = 0 \ , \quad [J_\pm, J_3] = 0 \ , \tag{33}$$

which is a particular q-deformed version $U_q(su(2) \oplus u(1))$ of the universal enveloping algebra of the *non-simple* algebra $su(2) \oplus u(1)$. This is in contrast with the well-known fact that the q-analog $U_q(su(2))$ of the (simple) algebra $su(2)$ is realizable in terms of two independent modes of the BM q-oscillator[1,2]. The Hopf-algebra structure of the q-deformed algebra (33) can be written out along the lines of Ref. 13, so we do not reproduce it here.

Using the q-Fock space framework, see (13)-(14) above, one is lead to the following rather simple yet q-dependent representation formulas ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...;$ $m = -j, -j+1, ..., j-1, j$) for the q-algebra $U_q(su(2) \oplus u(1))$:

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)q^{j-m-1}q^{j+m}} \ |j,m+1\rangle \\ &= q^{j-\frac{1}{2}} \sqrt{(j-m)(j+m+1)} \ |j,m+1\rangle \ , \end{aligned} \tag{34}$$

$$\begin{aligned} J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)q^{j+m-1}q^{j-m}} \ |j,m-1\rangle \\ &= q^{j-\frac{1}{2}} \sqrt{(j+m)(j-m+1)} \ |j,m-1\rangle \ , \end{aligned} \tag{35}$$

$$J_0 \ |j,m\rangle = m \ |j,m\rangle \ , \qquad\qquad J_3 \ |j,m\rangle = j \ |j,m\rangle \ . \tag{36}$$

Remark 2. As an interesting peculiarity of this quantum algebra, let us emphasize the following fact: at $q < 1$, due to the multiplier $q^{j-\frac{1}{2}}$ in formulas (34)-(35), the behavior of the matrix elements of the raising/lowering operators in the 'large spin' limit, i.e. for $j \to \infty$, principally differs from that of J_0 and J_3. Namely, at $j \to \infty$ we have $\langle j,m|J_3|j,m\rangle \to \infty$, unlike the limit: $\lim_{j\to\infty}\langle j,m\pm 1|J_\pm|j,m\rangle = 0$. The latter behavior is completely analogous to the (exponential) 'cutoff' property of the TD type q-oscillator whose energy values satisfy: $E_n \to 0$ when $n \to \infty$. On the other hand such asymptotics contrasts with large j behavior of representation matrix elements of J_+, J_- of the usual $su(2)$.

8. Concluding remarks

Our goal was to explore the rather unusual and seemingly overlooked properties of the TD type q-oscillator. In particular, we have analyzed various types of occurring degeneracies: nearest-neighbor levels, zero level E_0 & some other level E_m, $m \geq 2$, etc. The considered degeneracies are different from the two-fold degeneracies of oscillator-like system studied in Ref. 16 where due to extra cyclic symmetry, an infinite series of pairs of degenerate energy levels appears. In the case of TD q-oscillator, at a properly fixed value of the q-parameter the degeneracy is 'accidental' (without underlying symmetry) and involves single pair of levels.

For this, and other properties studied above, we expect for the TD q-oscillator a number of interesting physical applications. Let us mention the already proposed

usage[10] of TD q-oscillators as the base for the related "TD q-Bose gas" model, the $p=q$ one-parameter limit of q,p-Bose gas model. As argued[10], this TD-type q-Bose gas model gives as well efficient description of the observed non-Bose properties of the two- and multi-pion (-kaon) correlations in the experiments on relativistic heavy-ion collisions as the other, e.g. the BM, types of q-Bose gas model do.[17,18]

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